Let $G = \langle S \rangle$ be a finitely generated group, i.e. $|S| < \infty$, and any $g \in G$ can be written as

$$g = s_1 \cdots s_n, \quad s_i \in S \cup S^{-1}$$

Then we can define length function

$$|g|_S := \min\{n \geq 0 : \exists s_1, \dots, s_n \in S \cup S^{-1}, \; g = s_1 \cdots s_n\}$$

We can define a metric $d_S : G \times G \to \mathbb{R}_{\geq 0}$
$g, h \in G$, $d_S(g,h) := |g^{-1}h|_S$.

Say for now $|e_G|_S = 0$.

Lemma: (G, d_S) is a metric space.

Visualisation $\to$ Cayley graph $Cay(G,S)$ is the graph whose vertices are elements of G, and we connect $g, h \in G$ if $\exists s \in S \cup S^{-1}$ s.t. $h = gs$.

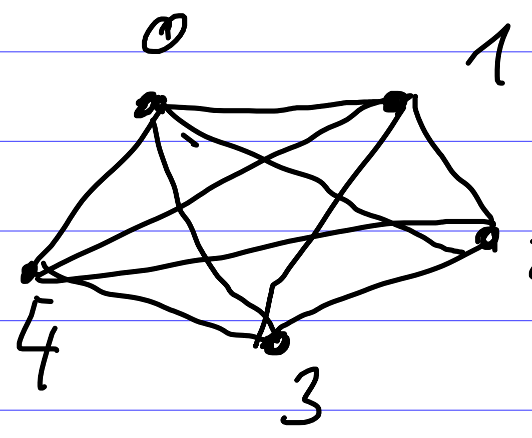
Observations: • As S generates G, $Cay(G,S)$ is connected.

• There is natural metric on $Cay(G,S)$, namely the shortest path metric.

Is this metric different from d_S ?

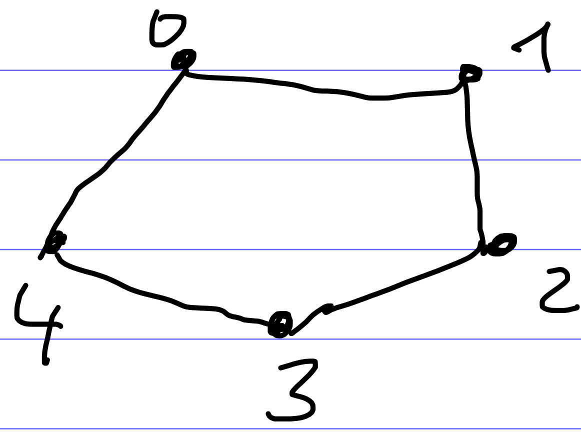
Lemma: (G, d_S), $(\mathrm{Cay}(G,S), d_{path})$ are isometric.

Examples: • $G = \mathbb{Z}/5\mathbb{Z}$, $S = G \setminus \{0\}$



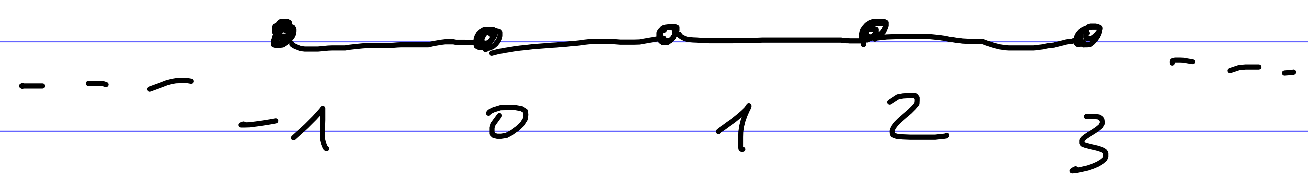
a complete graph

For G, $S' = \{1\} \rightarrow$



a 5-cycle.

• $G = \mathbb{Z}$, $S = \{1\}$:



Because of the previous lemma, we always identify a group with its Cayley graph

Lemma: Let G be a f.g. group, with 2 different finite generating sets $S, R \subset G$. There exist $A, B > 0$ such that

$$A \cdot |g|_R \leq |g|_S \leq B \cdot |g|_R \quad \forall g \in G.$$

Consequently:

$$A \cdot d_R(g,h) \leq d_S(g,h) \leq B \cdot d_R(g,h)$$

for any $g, h \in G$.

Proof: Let $g \in G$. Write $n := |g|_R$, i.e. $\exists r_1, \dots, r_n \in R \cup R^{-1}$, $g = r_1 \dots r_n$.

As S generates G, we can write

$$r_i := s_{i,1} s_{i,2} \dots s_{i,\ell_i}$$

Then $g = s_{1,1} s_{1,2} \dots s_{1,\ell_1} s_{2,1} \dots s_{2,\ell_2} \dots s_{n,1} \dots s_{n,\ell_n}$ and thus

$$|g|_S \leq \ell_1 + \dots + \ell_n \leq B \cdot n = B \cdot |g|_R$$

where $B := \max\{|r|_S : r \in R\}$. Similarly we have $A \cdot |g|_R \leq |g|_S$,

with $A = (\max\{|s|_R : s \in S\})^{-1}$. $\square$

Definition: Let X, Y be metric spaces. A map $\varphi: X \to Y$ is a quasi-isometry if $\exists C \geq 1, K \geq 0$ s.t:

(i) $\frac{1}{C} d_X(x,y) - K \leq d_Y(f(x), f(y)) \leq C\, d_X(x,y) + K$

for any $x, y \in X$.

(ii) $d_Y(y, f(X)) \leq K \quad \forall y \in Y$, i.e.

$\forall y \in Y, \exists x_y \in X, \ d_Y(y, f(x_y)) \leq K$.

Remark: Being quasi-isometric is an equivalence relation between metric spaces.

Examples: (0). Bounded metric spaces are all quasi-isometric.

(1) The map $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry: any $x \in \mathbb{R}$ is at dist. ≤ 1 from $\lfloor x \rfloor \in \mathbb{Z}$.

More generally, $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$, any $(x_1, \dots, x_n) \in \mathbb{R}^n$ is at distance $\leq \sqrt{n}$ of $(\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$

(2) For any finite group F and any f.g. G, G and $G \times F$ are quasi-isometric.

How to "easily" produce quasi-isometries?

Def: A metric space X is quasi-geodesic if there are $C > 0, K \geq 0$ s.t. any two points $x, y \in X$ are connected by a (C, K)-quasi-geodesic, i.e.

a (C,K)-quasi-isometric embedding

$$\gamma : [0, d(x,y)] \to X$$

s.t. $\gamma(0) = x$, $\gamma(d(x,y)) = y$.

Examples: • Any normed space $(V, \|\cdot\|)$ is geodesic: given $x, y \in V$, we can define $\gamma : [0,1] \to V$, $t \mapsto (1-t)x + ty$.

• Any connected graph with its path metric is quasi-geodesic.

Def: Let G be a group acting on a set X. The action is:

- cobounded if $\exists\, x_0 \in X$ and $R \geq 0$ s.t. any $x \in X$ is at dist. $\leq R$ from $G \cdot x_0$.
- metrically proper if there is $x_0 \in X$ such that $\{g \in G : d_X(x_0, gx_0) \leq R\}$ is finite for any $R \geq 0$.

Theorem (Milnor-Svarc lemma): Let G be a group acting isometrically on X, a quasi-geodesic metric space. If the action is cobounded and metrically proper, then G is finitely generated, and the map

$$\iota_{x_0} : G \longrightarrow X$$
$$g \mapsto g \cdot x_0$$

is a quasi-isometry for any $x_0 \in X$.

Corollary: Let G be a f.g. group. Let $H \leq G$ be a finite-index subgroup. Then H is f.g., and the inclusion map $H \hookrightarrow G$ is a quasi-isometry.

Proof: Let $G = \langle S \rangle$, $|S| < \infty$, and consider $H \curvearrowright (G, d_S)$. It is isometric.

Metric properness: Let $R \geq 0$

$$\{ h \in H : d_S(1_G, h) \leq R \} = H \cap \underbrace{B_{d_S}(1_G, R)}_{\text{finite}}$$

$\Rightarrow$ action is metrically proper.

Co-boundedness: $[G : H] < \infty$, so there are $a_1, \dots, a_r \in G$ s.t.

$$G = Ha_1 \sqcup \dots \sqcup Ha_r$$

Then, given $g \in G$, there is $i \in \{1, \dots, r\}$ s.t. $g \in Ha_i$, and thus

$$d_S(g, H \cdot 1_G) \leq |a_i|_S \leq \max_{1 \leq i \leq r} |a_i|_S$$

so the action is cobounded. Then Milnor - Švarc's lemma gives the result. □

Lemma: A metric space is quasi-geodesic if and if only it is quasi-isometric to a connected graph.

Proof: "⇐": obvious, since connected graphs are quasi-geodesic, and that quasi-geodesicity is preserved by quasi-isometries.

"⇒": let X be a quasi-geodesic metric space, i.e. any $x, y \in X$ are connected by a (C, K) - quasi-geodesic. Let $R := C + K$, and let Y be the graph whose vertices are elements of X, and whose edges connect $x, y \in X$ if $d_X(x,y) \leq R$.

Claims: Y is connected, and the natural $X \to Y$ is a quasi-isometry. $\square$

Our first invariant: the growth type of finitely generated groups.

Def: Let $G = \langle S \rangle$, $|S| < \infty$. The growth function of G w.r.t. S is

$$\gamma_{(G,S)} : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto \gamma_{(G,S)}(n) = |B_{d_S}(1_G, n)|.$$

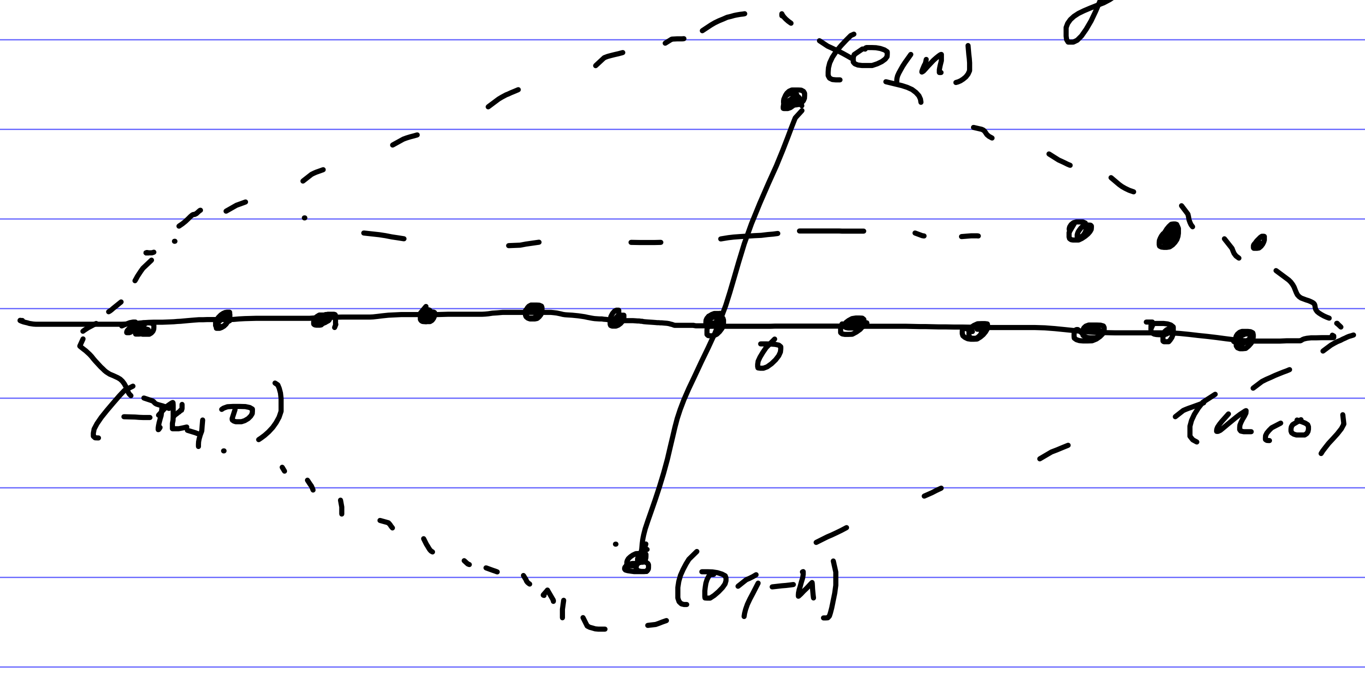
Examples: $G = \mathbb{Z}$, $S = \{\pm 1\}$. Then

$$\gamma_{(G,S)}(n) = |B(0,n)| = |\{-n, \dots, n\}| = 2n+1$$

- $G = \mathbb{Z}^2$, $S = \{\pm(1,0), \pm(0,1)\}$. Then

$$B((0,0), n) = \{(i,j) \in \mathbb{Z}^2 : |i| + |j| \leq n\}$$

i.e.



and then

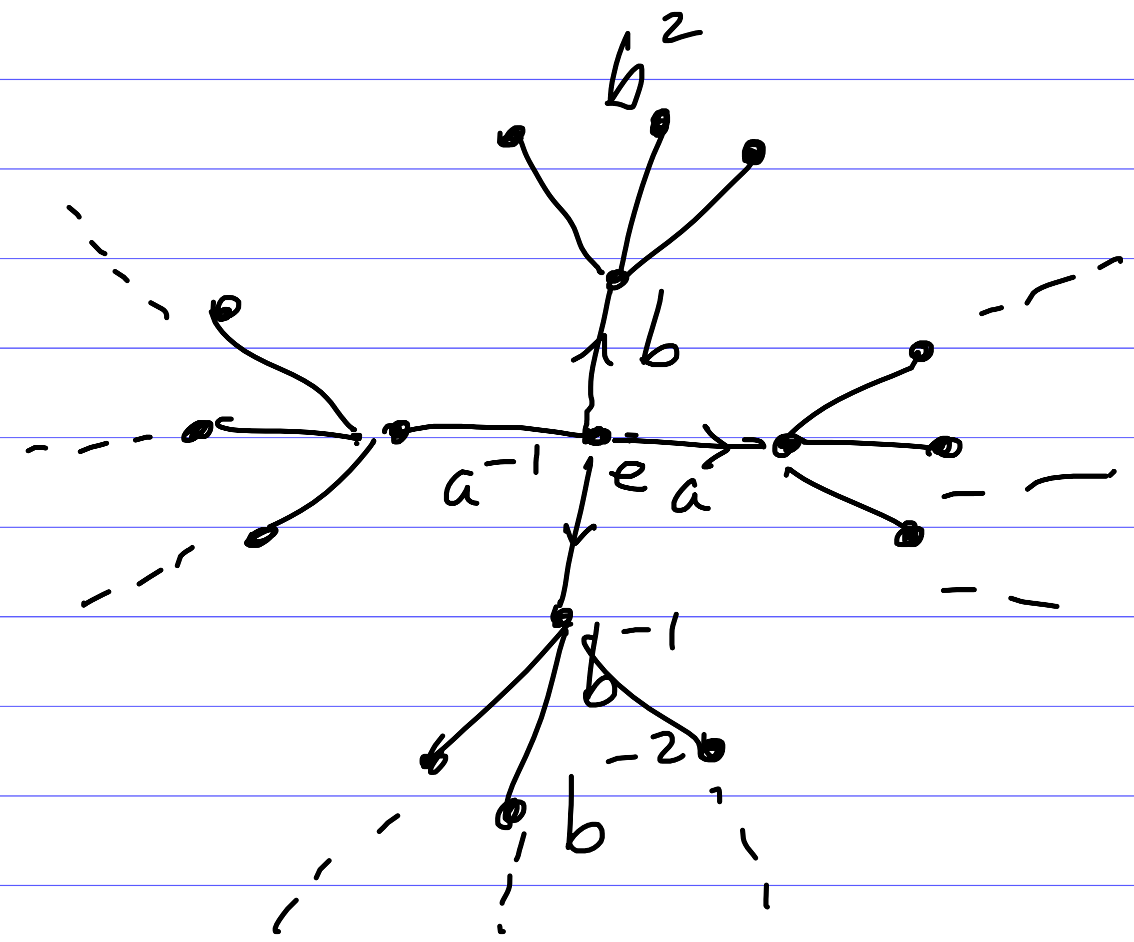
$$\gamma_{(\mathbb{Z}^2, S)}(n) = 2(1+3+5+\dots+(2n-1)) + (2n+1) = 2n^2+2n+1.$$

If rather $S' = S \cup \{(1,1), (-1,-1), (-1,1), (1,-1)\}$

then $B_{d_{S'}}((0,0), n) = \{-n, \dots, n\}^2$

$$\Rightarrow \gamma_{(G,S')}(n) = (2n+1)^2 = 4n^2+4n+1.$$

- $G = F_2 = \langle a, b \rangle$, $S = \{a^{\pm 1}, b^{\pm 1}\}$.



$$\gamma_{(F_2,S)}(n) = 1 + 4 \cdot \sum_{j=0}^{n-1} 3^j = 1 + 4 \cdot \frac{3^n - 1}{2} = 2 \cdot 3^n - 1.$$

Definition: Let $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ be two non decreasing maps. We say that ψ dominates φ, written $\varphi \precsim \psi$, if $\exists C > 0$, $\varphi(n) \leq C \cdot \psi(Cn)$ $\forall n \geq 0$.
We say that φ is asymptotically equivalent to ψ, written $\varphi \simeq \psi$, if $\varphi \precsim \psi$ and $\psi \precsim \varphi$.

Examples: • $n^a \precsim n^b \Leftrightarrow a \leq b$, and thus $n^a \simeq n^b \Leftrightarrow a = b$.

• $\forall a, b > 1$, $a^n \simeq b^n$.
• If $P(n)$ is a polynomial of degree $d \geq 1$, then $P(n) \simeq n^d$.
• $\forall d \geq 1$, $n^d \precsim e^n$, $n^d \not\simeq e^n$.

Theorem: If there exists a quasi-isometry $f: G \to H$, then

$$\gamma_{(G,S)} \simeq \gamma_{(H,T)}$$

where $G = \langle S \rangle$, $H = \langle T \rangle$.

Proof: Say that $f: G \to H$ is a (C,K)-Q.I.
First, let $g \in B_{d_S}(1_G, n)$. Then
$d_T(f(g), 1_H) \leq d_T(f(g), f(1_G)) + d_T(f(1_G), 1_H)$

$$\le C d_S(g, 1_G) + K + E \qquad =: E$$

$$\le C \cdot n + K + E$$

$$\lesssim (\underbrace{C+K+E}_{=:D}) \cdot n$$

so $\exists D \geq 0$, $f(B_{d_S}(1_G, n)) \subseteq B_{d_T}(1_H, D \cdot n)$

Claim: Pre-images of points under f have a uniformly bounded cardinality.

Proof: Let $y \in H$, $g, g' \in G$ s.t. $f(g) = y = f(g')$.

Then $d_S(g, g') \le C \cdot (d_T(f(g), f(g')) + K)$

$$= C \cdot K.$$

$\Rightarrow \exists F \geq 0$, $|f^{-1}(\{y\})| \le F$. $\square$

Putting this together:

$$\gamma_{(G,S)}(n) = |B_{d_S}(1_G, n)|$$

$$\le |f^{-1}(f(B_{d_S}(1_G, n)))|$$

$$= \Big| \bigcup_{y \in f(B_{d_S}(1_G, n))} f^{-1}(\{y\}) \Big|$$

$$\le F \cdot |f(B_{d_S}(1_G, n))|$$

$$\lesssim F \cdot |B_{d_T}(1_H, D \cdot n)|$$
$$= F \cdot \gamma_{(H,T)}(D \cdot n)$$
$$\leq Q \cdot \gamma_{(H,T)}(Qn)$$

where $Q := \max(D, F) > 0$.

Thus $\gamma_{(G,S)} \precsim \gamma_{(H,T)}$. By symmetry, we get that $\gamma_{(H,T)} \precsim \gamma_{(G,S)}$. $\square$

Corollary: If G is f.g., the asymptotic behaviour of the growth function is independent of the choice of the generating sets.

Corollary: For any $d \geq 2$, $\mathbb{Z}^d \not\simeq_{Q.I.} F_d$ and $\mathbb{Z}^d \not\simeq_{Q.I.} \mathbb{Z}^k$ for any $d \neq k$.

Definition: • G has polynomial growth of degree $d \geq 1$, if $\gamma_G \simeq n^d$.

- G has exponential growth if $\gamma_G \simeq e^n$.
- G has intermediate growth if $\gamma_G \succnsim n^d$ for any $d \geq 1$, and $\gamma_G \not\simeq e^n$.
- G has subexponential growth if $\gamma_G \precsim e^n$, and $\gamma_G \not\simeq e^n$.

Remark: Any f.g. group has at most exponential growth, because any such group is a quotient of a finite rank free group, and growth decreases when passing to quotients.

Important results:

Theorem (Milnor + Wolf): A finitely generated solvable group either has polynomial or exponential growth.

Theorem (Gromov): A f.g. group has polynomial growth if and only if it is virtually nilpotent.

$\gamma_0(G) = G$, $\gamma_1(G) = [G, G]$, $\gamma_2(G) = [\gamma_1(G), G]$, $\gamma_i(G) = [\gamma_{i-1}(G), G]$, ...

Theorem (Grigorchuk, 1981, 1984): Intermediate growth groups exist.

→ Erschler and Zheng gave the precise asymptotics behaviour of the growth function: e^{n^α}, where α is explicit.

Polycyclic groups

Definition: A group G is polycyclic if it has a sequence of subgroups
$$G = H_s \geq H_{s-1} \geq \dots \geq H_1 \geq H_0 = \{1_G\}$$
such that:

(i) $H_i \trianglelefteq H_{i+1}$, $i = 0, \dots, s-1$.

(ii) H_{i+1}/H_i is cyclic

Examples: • Any polycyclic group is solvable.

• Any finite solvable group is polycyclic.

• Any f.g. abelian group is polycyclic.

Definition: A group is polycyclic by finite if $\exists$ $G = H_s \geq H_{s-1} \geq \dots \geq H_1 \geq H_0 = \{1_G\}$ such that $H_i \trianglelefteq H_{i+1}$ $\forall i = 0, \dots, s-1$, and H_{i+1}/H_i is either cyclic or finite.

Proposition: Let G be a polycyclic-by-finite group. Then any subgroup of G is finitely generated.

Proof: Let

$$G = H_s \geq H_{s-1} \geq \dots \geq H_1 \geq H_0 = \{1_G\}$$

s.t. $H_i \trianglelefteq H_{i+1}$, $i=0,\dots,s-1$, H_{i+1}/H_i is either cyclic or finite. Note that $H_1,\dots,H_{s-1}$ are also polycyclic by finite. We prove the statement by induction on s. If $s=0$, there is nothing to show. Suppose that any subgroup of H_{s-1} is finitely generated. Let $H \leq H_s = G$. Then, $H \cap H_{s-1} \leq H_{s-1}$ is finitely generated, and

$$H/_{H \cap H_{s-1}} \cong HH_{s-1}/_{H_{s-1}} \leq H_s/_{H_{s-1}} = G/_{H_{s-1}}$$

and $G/_{H_{s-1}}$ is either finite or cyclic. In any case, it has all its subgroups finitely generated. Hence $H/_{H \cap H_{s-1}}$ is finitely generated. Thus, since

$$1 \to H \cap H_{s-1} \to H \to H/_{H \cap H_{s-1}} \to 1$$

we deduce that H is finitely generated. $\square$

Stability properties for polycyclic groups:

Proposition: Subgroups, quotients and extensions of polycyclic groups are polycyclic.

The statement about f.g. subgroups of polycyclic groups is false in general.

Example: Wreath products. Let A, B be two groups. Consider

$$A \wr B := \left(\bigoplus_B A \right) \rtimes B,$$

where B acts on $\bigoplus_B A$ by shifting coordinates:

$$(b \cdot f)(b') := f(b^{-1} b').$$

If A, B are f.g., then $A \wr B$ is finitely generated as well. If $A = \langle S_A \rangle$, $B = \langle S_B \rangle$, then

$$\{ f_a : a \in S_A \} \cup S_B$$

generates $A \wr B$, where for $a \in A$

$$f_a : B \to A, \quad b \mapsto \begin{cases} a & \text{if } b = 1_B \\ 1_A & \text{otherwise.} \end{cases}$$

On the other hand, $\bigoplus_B A$ is not f.g. if B is infinite.

Corollary: $A \wr B$ is not polycyclic if B is infinite.

Corollary: $A \wr B$ is solvable but not polycyclic if A is solvable and B is solvable and infinite.

Milnor's theorem: A f.g. solvable group of subexponential growth is polycyclic.

Proof: Let G be such a group. It is enough to show $[G,G]$ is finitely generated. If it holds then we conclude by induction on the derived length: since $[G,G]$ is solvable, f.g and of subexponential growth, it is polycyclic, and the quotient $G/[G,G]$ is polycyclic as well, so G is polycyclic since

$$1 \to [G,G] \to G \to G/[G,G] \to 1$$

Let us show then that $[G,G]$ is finitely generated.

$G/[G,G]$ is finitely generated and abelian, we can find a sequence of subgroups

$$G = H_{s+1} \geqslant H_s \geqslant \dots \geqslant H_1 \geqslant H_0 = [G,G]$$

s.t. $[G : H_s] < \infty$ and H_i/H_{i-1} is infinite cyclic. Note that H_s is finitely generated. Thus, we have the conclusion by applying iteratively the following claim:

Claim: Let G be a f.g. group of subexp. growth, with a normal subgroup H s.t. $G/H \cong \mathbb{Z}$. Then H is finitely generated.

Proof: Write $G/H = \langle aH \rangle$ and let $X \subseteq G$ be a finite symmetric generating set, which we may assume to contain a and a^{-1}. For any $x \in X$, there is $n \in \mathbb{Z}$ and $h \in H$ s.t. $x = a^n h$. By replacing any $x \in X$ by the corresponding h, we may suppose

$$X = \{a^{\pm 1}, h_1^{\pm 1}, h_2^{\pm 1}, \dots, h_\ell^{\pm 1}\}$$

with $h_i^{\pm 1} \in H$.

Let $H^{(m)} \subseteq H$ be the subgroup generated by

$$\{ a^j h_i^{\pm 1} a^{-j} ; \ i = 1, .., l, \ j = 0, ..., m \}.$$

Then $H^{(0)} \subseteq H^{(1)} \subseteq H^{(2)} \subseteq \dots$ and let

$$H^+ := \bigcup_{m \geq 0} H^{(m)}.$$

We show that $H^+ = H^{(m)}$ for some $m \geq 1$.

If not, for any $m \geq 1$, there is $j_m \in \{1, .., l\}$ such that

$$k_m := a^m h_{j_m} a^{-m} \in H^{(m)} \setminus H^{(m-1)}.$$

Now, for $m \in \mathbb{N}$, consider the products

$$k_0^{\varepsilon_0} k_1^{\varepsilon_1} \dots k_m^{\varepsilon_m}, \quad \varepsilon_i \in \{0, 1\}$$

There are 2^{m+1} words of this type which represent distinct group elements of length $\leq 3m+1$ in the generating set X. Indeed the maximal length is attained when $\varepsilon_i = 1$ for all i, and the corresponding word is

$$k_0 k_1 \dots k_m = h_{j_0} (a h_{j_1} a^{-1})(a^2 h_{j_2} a^{-2}) \dots (a^m h_{j_m} a^{-m})$$

$$= \underbrace{h_{j_0}}_{\text{length}=1} \underbrace{a h_{j_1} a h_{j_2} a \dots a h_{j_m}}_{\text{length} \leq 2m} \underbrace{a^{-m}}_{\text{length} \leq m}$$

Hence $\gamma_{(G,X)}(3m+1) \geq 2^{m+1}$ so

$$\gamma_{(G,X)}(n) \succsim 2^n$$

a contradiction with the fact that G has subexponential growth. Thus $H^+ = H^{(m')}$ for some $m' \geq 1$, and hence H^+ is f.g.

Similarly, $H^- := \bigcup_{m=0}^{\infty} H^{(-m)}$, where $H^{(-m)}$ is generated by $a^{-j} h_i^{\pm 1} a^{j}$, $i = 1, \dots, \ell$, $j = 0, \dots, m$, is finitely generated and $H^- = H^{(-m'')}$ for some $m'' \geq 1$. Thus

$$\begin{aligned} H = \left\langle \bigcup_{j=-\infty}^{\infty} H^{(j)} \right\rangle &= \langle H^+ \cup H^- \rangle \\ &= \langle H^{(m')} \cup H^{(-m'')} \rangle \end{aligned}$$

$\Rightarrow$ H is finitely generated. $\square$ Claim.

$h \in H$: $\quad h = a^{\varepsilon_1} \underbrace{h_{i_1}^{n_1}} a^{\varepsilon_2} \underbrace{h_{i_2}^{n_2}} \dots a^{\varepsilon_n}$

$$\downarrow \pi \qquad\qquad\qquad \swarrow \pi$$

$$1_{G/H} = a^{\varepsilon_1 + \dots + \varepsilon_n}$$

What about the quasi-isometric rigidity of wreath products?

Lemma (Erschler, 2000): If A, B are bilipschitz equivalent, then $A \wr G$ and $B \wr G$ are bilipschitz equivalent, for any G f.g.

Corollary: Virtual solvability is not a Q.I. invariant.

Proof: Let $A = \mathbb{Z}$, $B = \mathbb{Z} \times A_5$, $G = \mathbb{Z}$. By the previous lemma, $A \wr G$ and $B \wr G$ are quasi-isometric, and $A \wr G$ is solvable, and $B \wr G$ is not. $\square$

Open question: Is virtual solvability a Q.I. invariant among finitely presented groups?

Theorem (Eskin-Fisher-Whyte, 2012, 2013): Let F, F' be two non-trivial finite groups.

Then $F \wr \mathbb{Z}$ and $F' \wr \mathbb{Z}$ are quasi-isometric if and only if there exist $d, r, s \geq 1$

such that $|F| = d^r$, $|F'| = d^s$.

Theorem (Genevois – Tessera, 2024): Let $n, m \geq 2$. Let G, H be finitely presented one ended groups.

- If G is not amenable, then $\mathbb{Z}/n\mathbb{Z} \wr G$ and $\mathbb{Z}/m\mathbb{Z} \wr H$ are quasi-isometric if and only if n, m have the same prime divisors and G and H are quasi-isometric.
- If G is amenable, then $\mathbb{Z}/n\mathbb{Z} \wr G$ and $\mathbb{Z}/m\mathbb{Z} \wr H$ are quasi-isometric if and only if there exist $d, r, s \geq 1$ s.t. $n = d^r$, $m = d^s$ and there exists a quasi-$\frac{s}{r}$-to-one quasi-isom. $G \to H$.